ISSN 1064-5624, Doklady Mathematics, 2013, Vol. 88, No. 2, pp. 532–536. © Pleiades Publishing, Ltd., 2013.
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MATHEMATICS

On Distance-Regular Graphs in Which Neighborhoods of Vertices Are Strongly Regular

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Received February 27, 2013

DOI: 10.1134/S1064562413050116

We consider undirected graphs without loops or multiple edges. Given a vertex a in a graph Γ, let $\Gamma_i(a)$ denote the i-neighborhood of a, i.e., the subgraph induced by Γ on the set of all its vertices that are a distance of i away from a. Let $[a] = \Gamma_1(a)$ and $a^\perp = \{a\} \cup [a]$.

A strongly regular graph with parameters (v, k, λ, μ) and with integer eigenvalues k, r, and s, where $s < 0$, is called a Smith graph if

$$k = \frac{-s((2r+1)(r-s)-r(r+1))}{(r-s)+r(r+1)},$$

$$l = \frac{-(s+1)((2r+1(r-s)-r(r+1))}{(r-s)-r(r+1)},$$

$$\lambda = \frac{-r(s+1)((r-s)-r(r+3))}{(r-s)+r(r+1)},$$

$$\mu = \frac{-(r+1)s((r-s)-r(r+1))}{(r-s)+r(r+1)}.$$

If vertices u and w are separated by a distance of i in Γ, then $b_i(u, w)$ ($c_i(u, w)$) denotes the number of vertices in the intersection of $\Gamma_{i+1}(u)$ ($\Gamma_{i-1}(u)$) with $[w]$. A graph Γ of diameter d is called a distance-regular graph with the intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ if the values $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of the vertices u and w separated by a distance of i in Γ for any $i = 0, 1, \ldots, d$. Let $a_i = k - b_i - c_i$. The Taylor graph is a distance-regular graph with the intersection array $\{k, \mu, 1; 1, \mu, k\}$ (see [1]).

Let Γ be a distance-regular graph of diameter $d \geq 3$, and let $\theta_0 > \theta_1 > \ldots > \theta_d$ be the eigenvalues of Γ. According to [2], we have the fundamental bound

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) \geq -\frac{ka_1b_1}{(a_1+1)^2}.$$

Let

$$b^+ = -1 - \frac{b_1}{1+\theta_d}, \quad b^- = -1 - \frac{b_1}{1+1+\theta_1}.$$

A nonbipartite graph for which the fundamental bound becomes an equality is called tight. The neighborhood of any vertex in a tight graph is a strongly regular graph with eigenvalues a_1, b^+, b^-. It is well known that a tight graph of diameter 3 is a Taylor graph (see, e.g., [2, Theorem 3.2]). In this case, the neighborhood of any vertex is a strongly regular graph with $k' = 2\mu'$.

An incidence system with a set of points P and a set of straight lines $\mathcal{L}$ is called an α-partial geometry of order (s, t) (denoted by $pG_\alpha(s, t)$) if each line contains $s + 1$ points; each point lies on $t + 1$ lines; any two points lie on at most one line; and, for any antiflag $(a, l) \in (P, \mathcal{L})$, there are precisely α lines that pass through a and intersect l. If $\alpha = 1$, the geometry is called a generalized quadrangle and is denoted by $GQ(s, t)$. The point graph of a geometry is defined on the set of points P, and two points are adjacent if they lie on a line. The point graph of $pG_\alpha(s, t)$ is strongly regular with $v = (s+1)\left(1 + \frac{st}{\alpha}\right)$, $k = s(t+1)$, $\lambda = s - 1 + t(\alpha - 1)$, and $\mu = \alpha(t+1)$. A strongly regular graph with such parameters for some positive integers α, s, and t is called a pseudogeometric graph for $pG_\alpha(s, t)$.

Makhnev has proposed a program for the study of distance-regular graphs in which the neighborhoods of vertices are strongly regular graphs with given parameters. This program has been implemented in the case of strongly regular graphs with the eigenvalue 2 (see [3]).

An antipodal distance-regular graph Γ of diameter 3 has (see [1]) the intersection array $\{k, \mu(r-1), 1; 1, \mu, k\}$ and possesses $v = r(k+1)$ vertices and the spectrum $k^1, n^f, (-1)^k, (-m)^h$, where n and $-m$ are the roots of the equation $x^2 + (\mu - \lambda)x - k = 0$, $f = \frac{m(r-1)(k+1)}{n+m}$, and $h = \frac{n(r-1)(k+1)}{n+m}$.

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If $\mu \neq \lambda$, then the eigenvalues of the graph are integer and the parameters of the graph can be expressed in terms of r, n, and m: $k = nm$, $\mu = \frac{(m-1)(n+1)}{r}$, and $\lambda = \mu + n - m$. Since the multiplicities of the eigenvalues are integer-valued, we have the following divisibility condition: $n + m$ divides $(r-1)m(m^2-1)$. If $r > 2$, the Krein condition $q_{33}^3 \geq 0$ implies $m \leq n^2$. If $m = n^2$, then the neighborhood of any vertex is a strongly regular graph with eigenvalues $a_1 = (n-1)\left(\frac{(n+1)^2}{r} - n\right)$, $n - \frac{n+1}{r}$, and $n - \frac{(n+1)^2}{r}$. A graph with $m = n^2$ (equivalently, with $q_{33}^3 = 0$) is called a Krein graph of diameter 3.

Theorem 1. *Let* $n + 1 = ru$ *and* Γ *be a Krein graph of diameter* 3 *with the intersection array* $\{n^3, (r-1)u(n^2-1), 1; 1, u(n^2-1), n^3\}$. *Then the following assertions hold*:

(1) *For any vertex* $a \in \Gamma$, *the subgraph* $\Delta = [a]$ *is a pseudogeometric graph for* $pG_{u-1}(ru-2, ru^2-ru)$ *with eigenvalues* $(ru-2)(ru^2-ru+1)$, $ru-1-u$, *and* $-(ru^2-ru+1)$.

(2) *If n is a power of a prime number, then there exist graphs* Γ *and* Δ. *Specifically*,

(i) *If* $r = \frac{n+1}{2}$, *then there exists a graph* Γ *that is locally a* $GQ(n-1, n+1)$*-graph*.

(ii) *There exist pseudogeometric graphs for* $pG_2(4, 12)$, $pG_3(6, 24)$, $pG_2(7, 18)$, $pG_4(8, 40)$, $pG_2(10, 24)$, $pG_3(10, 36)$, $pG_5(10, 60)$, *and* $pG_6(12, 84)$.

Let Γ be an antipodal graph of diameter 4 and $\bar{\Gamma}$ be the antipodal quotient of Γ. Then, by [1, Proposition 4.2.2], Γ has the intersection array $\{k, k - a_1 - 1, (r-1)c_2, 1; 1, c_2, k - a_1 - 1, k\}$. If the Krein parameters vanish, we have the following three interesting cases: $q_{11}^4 = 0$, $q_{44}^4 = 0$, and $q_{11}^4 = q_{44}^4 = 0$.

The equality $q_{44}^4 = 0$ implies that $\bar{q}_{22}^2 = 0$ in $\bar{\Gamma}$ and $\bar{\Gamma}$ is a Smith graph. Therefore, the neighborhoods of vertices in Γ (which are isomorphic to the neighborhoods of vertices in $\bar{\Gamma}$) are strongly regular. A graph with $q_{44}^4 = 0$ is called a Krein graph of diameter 4. The following result concerning Krein graphs of diameter 4 is due to Jurisic [4].

Proposition 1. *Let* Γ *be a Krein graph of diameter* 4. *Then one of the following assertions holds*:

(1) $\bar{\Gamma}$ *is a Smith graph without triangles, and* Γ *has the intersection array* $\{k, k-\mu, (r-1)\mu, 1; 1, \mu, k-1, k\}$, *where* $k = t^3 + 3t^2 + t$ *and* $r\mu = t(t+1)$.

(2) $\bar{\Gamma}$ *is a Smith graph with* $a_1 > 0$ *and nonprincipal eigenvalues t and s*; *and the nonprincipal eigenvalues of the constituents of* $\bar{\Gamma}$ *are equal to* t, $s_1 = \frac{t^2+2t+s}{2}$, t, $s_2 = \frac{s-t^2}{2}$ *and either*

(i) $t = q - 2$ *and* Γ *is a tight graph* AT4$(q-2, q, r)$ *with the intersection array* $\left\{q(q^2-2), (q^2-1)(q-1), \frac{(r-1)\cdot 2q(q-1)}{r}, 1; 1, \frac{2q(q-1)}{r}, (q^2-1)(q-1), q(q^2-2)\right\}$;

(ii) Γ *is a cover of the point graph of the generalized quadrangle* $GQ(w, w^2)$, $w = t + 1$ *is odd, and* $4w^3 + (w+1)^2$ *is a square; or*

(iii) *s and t are even*, $q = t + 2$ *and* $u = \frac{s+q^2}{2}$ *satisfy* $\sqrt{q} \leq u < q - 1$, $q(q-1) - u$ *divides* $(q-2)^2(q-1)^2q$, *and the expression* $4u^4(q-2)^2 - 4u^3(q-1)(q^3-q^2-6q+10) + u^2(q-1)^2(q^4+6q^3-15q^2+12q+20) - 2u(q-1)^3(q(q+1)(q^2+4)) + q^2(q-1)^4(q+2)^2$ *is a square*.

We show that case (2)(ii) is not possible. Let $w = 2l + 1$. Then $4w^3 + (w+1)^2 = 4(8l^3 + 12l^2 + 6l + 1) + 4(l^2 + 2l + 1)$. Therefore, $8l^3 + 13l^2 + 8l + 2$ is a square. If l is odd, then $8l^3 + 13l^2 + 8l + 2$ is congruent to 3 modulo 4. If l is even, then $8l^3 + 13l^2 + 8l + 2$ is congruent to 2 modulo 4. In either case, we obtain a contradiction.

By Theorem 5.2 in [2], Γ is tight if and only if $q_{11}^4 = 0$. If Γ is a tight graph with a neighborhood of a vertex having nonprincipal eigenvalues $p = b^+$, $-q = b^-$, then all the parameters of Γ can be expressed in terms of p, q, and r. In this case, Γ is said to be an antipodal tight graph of diameter 4 with parameters p, q, r (AT4(p, q, r)-graph). The results obtained by Koolen and Jurisic [2, 5] for AT4-graphs are stated as the following proposition.

Proposition 2. *Let* Γ *be an* AT4(p, q, r)*-graph. Then* Γ *is a graph with the intersection array* $\left\{q(pq+p+q), (q^2-1)(p+1), \frac{(r-1)q(p+q)}{r}, 1; 1, \frac{q(p+q)}{r}, (q^2-1)(p+1), q(pq+p+q)\right\}$ *and the following assertions hold*:

(1) *If* $p = 1$ *or* $r = p + q$, *then* Γ *is a Conway–Smith graph with* $(p, q, r) = (1, 2, 3)$.

(2) *If* $q = 2$, *then* Γ *is a Johnson graph* $J(8, 4)$ *with* $(p, q, r) = (2, 2, 2)$, *a half-8-cube with* $(p, q, r) = (4, 2, 2)$, *or a Conway–Smith graph*.

(3) *If* $(p, q, r) = (qs, q, q)$, *then* Γ *is a Johnson graph* $J(8, 4)$, *a half-8-cube*, *a* $3O_6^-(3)$*-graph with* $(p, q, r) = (3, 3, 3)$, *the second Meixner graph with* $(p, q, r) = (8, 4, 4)$, *or a* $3O_7(3)$*-graph with* $(p, q, r) = (9, 3, 3)$.

The central result in this work is the following one, which was obtained by Makhnev.

Theorem 2. *There do not exist* AT4$(q - 2, q, 2)$*-graphs*.

In view of the remark on case (2)(ii) in Proposition 1, we have the following result.

Theorem 3. *Let* Γ *be a Krein graph of diameter* 4. *Then one of the following assertions holds*:

(1) $\bar{\Gamma}$ *is a Smith graph without triangles, and* Γ *has the intersection array* $\{k, k - \mu, (r - 1)\mu, 1; 1, \mu, k - 1, k\}$, *where* $K = t^3 + 3t^2 + t$ *and* $r\mu = t(t + 1)$.

(2) $\bar{\Gamma}$ *is a Smith graph with* $a_1 > 0$ *and nonprincipal eigenvalues* t *and* s; *and the nonprincipal eigenvalues of the constituents of* $\bar{\Gamma}$ *are equal to* t, $s_1 = \dfrac{t^2 + 2t + s}{2}$, t, $s_2 = \dfrac{s - t^2}{2}$, *and either*

(i) $t = q - 2$, Γ *is a tight graph* AT4$(q - 2, q, r)$ $(2 < r \le q - 1)$ *with the intersection array* $\left\{q(q^2 - 2), (q^2 - 1)(q - 1), \dfrac{(r - 1)\cdot 2q(q - 1)}{r}, 1; 1, \dfrac{2q(q - 1)}{r}, (q^2 - 1)(q - 1), q(q^2 - 2)\right\}$ *and, for any vertex* a *of* Γ, *the subgraph* $\Gamma_2(a)$ *is an antipodal distance-regular graph of diameter* 4 *with the intersection array* $\left\{(q - 2)q^2, (q - 1)^3, \dfrac{2(r - 1)(q - 1)(q - 2)}{r}, 1; 1, \dfrac{2(q - 1)(q - 2)}{r}, (q - 1)^3, (q - 2)q^2\right\}$ *or*

(ii) s *and* t *are even and, for* $q = t + 2$ *and* $u = \dfrac{s + q^2}{2}$, *we have* $\sqrt{q} \le u < q - 1$, $q(q - 1) - u$ *divides* $(q - 2)^2(q - 1)^2q$, *and the expression* $4u^4(q - 2)^2 - 4u^3(q - 1)(q^3 - q^2 - 6q + 10) + u^2(q - 1)^2(q^4 + 6q^3 - 15q^2 + 12q + 20) - 2u(q - 1)^3(q(q + 1)(q^2 + 4)) + q^2(q - 1)^4(q + 2)^2$ *is a square*.

The following result holds for AT4-graphs.

Theorem 4. *Let* Γ *be an* AT4(p, q, r)*-graph*, u *be a vertex of* Γ, *and* $\Delta = [u]$. *Then the following assertions hold*:

(1) *If* $p = \alpha q$, *then* Δ *is a pseudogeometric graph for* $pG_\alpha(\alpha(q + 1), q - 1)$, $\alpha + 1$ *divides* $2(q^2 - 1)$, $\alpha + q$ *divides* $q(q^2 - 1)(q^2 + g - 1)(q + 2)$, $\dfrac{q^3\alpha(\alpha + 1)}{r}$ *is even*, $r(p + 1) \le q(p + q)$, *and* r *divides* $q(\alpha + 1)$.

(2) *If* Δ *is a graph without triangles*, *then* Γ *is a Conway–Smith graph or the first Soicher graph with* $(p, q, r) = (2, 4, 3)$.

(3) *If* $q = p + 2$, *then*

(i) $\dfrac{2p(p + 1)(p + 2)}{r}$ *is even*, $2 < r < p + 2$, r *divides* $2(p + 1)$, *and* Γ *is a graph with the intersection array*

$$\left\{(p+1)(p+2)^2, (p+3)(p+1)^2, \frac{(r-1)\cdot 2(p+1)(p+2)}{r};\ 1, 1, \frac{2(p+1)(p+2)}{r}, (p+3)(p+1)^2, (p+1)(p+2)^2\right\}.$$

(ii) *The antipodal quotient* $\bar{\Gamma}$ *is a graph with parameters* $\left(\dfrac{(p + 1)^2(p + 4)^2}{2}, (p + 2)(p^2 + 4p + 2, p(p + 3), 2(p + 1)(p + 2)\right)$ *and eigenvalues* p, $-(p^2 + 4p + 4)$.

(iii) *The second neighborhood of a vertex in* $\bar{\Gamma}$ *is a strongly regular graph with parameters* $((p + 1)(p + 3)(p^2 + 4p + 2), p(p + 2)^2, p^2 + p - 2, 2p(p + 1))$ *and eigenvalues* p, $-(p^2 + 2p + 2)$ *that has a distance-regular* r*-covering with the intersection array* $\left\{p(p + 2)^2, (p + 1)^3, \dfrac{2(r - 1)p(p + 1)}{r}, 1; 1, \dfrac{2p(p + 1)}{r}, (p + 1)^3, p(p + 2)^2\right\}$.

(4) *If* $p \le 4$, *then either*

(i) Γ *is a Conway–Smith graph or a Soicher graph with* $(p, q, r) = (2, 4, 3)$,

(ii) Γ *is a* $3O_6^-(3)$*-graph or* $p = 3$, $q = 5$, $r = 4$, *and* Γ *has the intersection array* $\{115, 96, 30, 1; 1, 10, 96, 115\}$, *while* $\Gamma_2(u)$ *has the intersection array* $\{75, 64, 18, 1; 1, 6, 64, 75\}$, *or*

(iii) Γ *is a half-8-cube or* $(p, q, r) = (4, 4, 2)$ *and* Γ *has the intersection array* $\{96, 75, 16, 1; 1, 16, 75, 96\}$, *or* $(p, q, r) = (4, 6, 5)$ *and* Γ *is a graph with the intersection array* $\{204, 175, 48, 1; 1, 12, 175, 204\}$ *and* $\Gamma_2(u)$ *is a graph with the intersection array* $\{144, 125, 32, 1; 1, 8,125, 144\}$.

(5) *If* $q \le 4$, *then either*

(i) Γ *is a Johnson graph* $J(8, 4)$, *a half*-8-*cube, or a Conway–Smith graph*,

(ii) Γ *is a* $3O_6^-(3)$-*graph*, $3O_7(3)$-*graph, or* $(p, q, r) = (p, 3, 2)$, Γ *is a graph with the intersection array* $\left\{3(4p+3), 8(p+1), \frac{3(p+3)}{2}, 1; 1, \frac{3(p+3)}{2}, 8(p+1), 3(4p+3)\right\}$ *and* $p = 9, 21$, *or*

(iii) Γ *is a half*-8-*cube, the second Meixner graph, the first Soicher graph, or* $r = 3$, Γ *is a graph with the intersection array* $\left\{4(5p+4), 15(p+1), \frac{8(p+4)}{3}, 1; 1, \frac{8(p+4)}{3}, 15(p+1), 4(5p+4)\right\}$ *and* $p = 8, 20, 44, 56, 116$, *or* $r = 2$, Γ *is a graph with the intersection array* $\{4(5p+4), 15(p+1), 4(p+4), 1; 1, 4(p+4), 15(p+1), 4(5p+4)\}$ *and* $p = 8, 9, 20, 21, 44, 56, 116$.

Let $n + 1 = ru$, Γ be a Krein graph of diameter 3 with the intersection array $\{n^3, (r-1)u(n^2-1), 1; 1, u(n^2-1), n^3\}$, a be a vertex of Γ, and $\Delta = [a]$. It was proved in [6, Lemma 5.3] that Δ is a strongly regular graph with parameters $v = n^3$, $k = (n-1)\left(\frac{(n+1)^2}{r} - n\right)$, $\lambda = r\left(\frac{n+1}{r} - 1\right)^3 + r - 3$, $\mu = \left(\frac{n+1}{r} - 1\right)\left(\frac{(n+1)^2}{r} - n\right)$ and with eigenvalues k, $n - \frac{n+1}{r}$, $n - \frac{(n+1)^2}{r}$. Therefore, Δ is a pseudogeometric graph for $pG_{u-1}(ru-2, ru^2-ru)$ with eigenvalues $k = (ru-2)(ru^2-ru+1)$, $ru-1-u$, $ru-1-ru^2$. The integer-valuedness assumption, i.e., $(u-1)(ru^2-u)$ divides $(ru-2)(ru-1)(ru^2-ru)(ru^2-ru+1)$, holds true.

Lemma 1. *If n is a power of a prime number, then the following assertions hold*:

(1) *If* $r = \frac{n+1}{2}$, *then there exists a graph* Γ *that is locally a* $GQ(n-1, n+1)$-*graph.*

(2) *There exist pseudogeometric graphs for* $pG_2(4, 12)$, $pG_3(6, 24)$, $pG_2(7, 18)$, $pG_4(8, 40)$, $pG_2(10, 24)$, $pG_3(10, 36)$, $pG_5(10, 60)$, *and* $pG_6(12, 84)$.

Proof. If n is a power of a prime number, then, in view of the remark presented after Lemma 5.2 in [6], Γ exists; therefore, Δ exists as well. If $r = \frac{n+1}{2}$, in view of [7], there exists a graph Γ that is locally a $GQ(n-1, n+1)$-graph.

For graphs with $n \le 13$, pseudogeometric graphs exist for $GQ(4, 6)$ and $pG_2(4, 12)$ ($n = 5$); for $GQ(6, 8)$ and $pG_3(6, 24)$ ($n = 7$); for $pG_2(7, 18)$ ($n = 8$); for $GQ(8, 10)$ and $pG_4(8, 40)$ ($n = 9$); for $GQ(10, 12)$, $pG_2(10, 24)$, $pG_3(10, 36)$, and $pG_5(10, 60)$ ($n = 11$); and for $GQ(12, 14)$ and $pG_6(12, 84)$ ($n = 13$). Lemma 1 and Theorem 1 are proved.

Lemma 2. AT4$(q-2, q, 2)$-*graphs do not exist.*

Proof. Let Γ be an AT4$(q-2, q, 2)$-graph with the intersection array $\{q(q^2-2), (q^2-1)(q-1), q(q-1), 1; 1, q(q-1), (q^2-1)(q-1), q(q^2-2)\}$. Then the neighborhood of a vertex u in Γ is a strongly regular graph with parameters $(q^3-2q, (q+1)(q-2), q-4, q-2)$ and, by Theorem 5.8 in [4], for any two vertices y and z separated by a distance at most 2 in $\Gamma_2(u)$, we have $|[u] \cap [y] \cap [z]| = q-1$. From this, for any two vertices $w, w' \in [u]$, it is true that $|[w] \cap [w'] \cap \Gamma_2(u)| = q^2 - 2q + 1$.

Thus, the pair $([u], \Gamma_2(u))$ is a quasi-symmetric $(q(q^2-2), (q^2-2)(q^2-1), (q^2-1)(q-1), q(q-1), q^2-2q+1)$-scheme in which any two blocks intersect in 0 or $q-1$ points.

By [8, Theorem 5.3], a block graph of a scheme is strongly regular with eigenvalues $\frac{(r-1)k - xb - y}{y - x} = q^2(q^2-q-1)-1$, q^2-2q, $-q$ and parameters $((q^2-2)(q^2-1), q^2(q^2-q-1)-1, q^4-2q^3+2q^2-3q-1, q^4-2q^3+q^2-1)$. The multiplicity of the eigenvalue q^2-2q is $\frac{q^2(q^2-q-1)-1)(q^2(q^2-q-1)+q-1)}{q(q^4-2q^3+q^2-1)}$, which contradicts the fact that the numerator of the fraction is not divided by q. The lemma and Theorem 2 are proved.

Lemma 3. *Let* Γ *be an* AT4(p, q, r)-*graph*, u *be a vertex of* Γ, *and* $\Delta = [u]$. *If* $p = 3$, *then either* $q = r = 3$ *and* Γ *is a* $3O_6^-(3)$-*graph or* $q = 5$, $r = 4$, Δ *has parameters* $(115, 18, 1, 3)$, *and* Γ *has the intersection array* $\{115, 96, 30, 1; 1, 10, 96, 115\}$.

Proof. Let $p = 3$. Then Δ has parameters $(4q^2+3q, 3(q+1), 6-q, 3)$. Furthermore, $q+3$ divides $q^2(q^2-1)$ and q^2+3 divides $q^2(q^2-1)(q^2+q-1)(q+2)$.

If $q = 3$, then Δ has parameters $(45, 12, 3, 3)$. In this case, $r = 3$ and Γ is a $3O_6^-(3)$-graph.

If $q = 5$, then Δ has parameters $(115, 18, 1, 3)$ and eigenvalues 3 and -5. By Lemma 2, the case $r = 2$ is not possible. Therefore, $r = 4$ and Γ has the intersection array $\{115, 96, 30, 1; 1, 10, 96, 115\}$.

If $q = 6$, then Δ has parameters $(162, 21, 0, 3)$ and eigenvalues 3 and -6. In this case, $q^2 + 3 = 39$ does not divide $q^2(q^2-1)(q^2+q-1)(q+2)$.

The cases $p = 4$ and $q = 3, 4$ are treated in a similar manner.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (project no. 12-01-00012 and joint project no. 12-01-91155 with the National Science Fund of China), by the Branch of Mathematics of the Russian Academy of Sciences (project no. 12-T-1-1003), and by the Ural Branch of the Russian Academy of Sciences jointly with the Siberian Branch of the Russian Academy of Sciences (project no. 12-S-1-1018) and with the National Academy of Sciences of Belarus (project no. 12-S-1-1009).

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Translated by I. Ruzanova